

# Variants of Bell inequalities

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A family of Bell-type inequalities is present, which are constructed directly from the “standard” Bell inequalities involving two dichotomic observables per site. It is shown that the inequalities are violated by all the generalized Greenberger-Horne-Zeilinger states of multiqubits. Remarkably, our new inequalities can provide stronger non-locality tests in a sense that the local reality inequalities are exponentially stronger than the corresponding multipartite separability inequalities. This reveals that the exponential violation of local realism by separable states is an interesting consequence of quantum fluctuation of multipartite systems.

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Bell's inequality [1] was originally designed to rule out various kinds of local hidden variable theories based on Einstein, Podolsky, and Rosen's (EPR's) notion of local realism [2], and thus certifies quantum origin and true nonlocality of entangled states [3]. Bell inequalities now serve a dual purpose. On one hand these inequalities provide a test to distinguish entangled from non-entangled quantum states. Indeed, experimenters routinely use violations of Bell inequalities to check whether they have succeeded in producing entangled states [4]. On the other hand, violations of the inequalities are applied to realizing certain tasks by quantum information, such as building quantum protocols to decrease communication complexity [5] and making secure quantum communication [6]. Derivations of new and stronger Bell-type inequalities are thus one of the most important and challenging subject in quantum theory.

There are extensive earlier works on Bell inequalities [7], including Clauser-Horne-Shimony-Holt (CHSH) inequality for bipartite systems [8] and Mermin-Roy-Singh-Ardehali-Belinskii-Klyshko (MK) inequalities for multi-particle systems [9]. A set of multipartite Bell inequalities with two dichotomic observables per site, including the MK inequalities, has been constructed by Werner and Wolf and by Żukowski and Brukner (WWZB) [10], which is complete in the sense that the inequalities are satisfied if and only if the correlations considered are describable in a local and realistic picture. Such inequalities are referred usually as "standard" ones. There are also other Bell-type inequalities with more than two observables per site and/or for high dimensional systems [11], quadratic Bell inequalities [12], and ones based on variances and Wigner-Yanase skew information [13]. We refer to [14] and references therein for more details.

The theorem of Gisin [15] states that any pure entangled state of two qubits violates a CHSH inequality. Can Gisin's theorem be generalized to multiqubits? Scarani and Gisin [16] show that there are states which do not violate the MK inequalities. Surprisingly, Żukowski *et al* [17] further show that there is a family of pure entangled states of  $n > 2$  qubits which do not violate any standard Bell inequality. This family is a subset of the generalized GHZ states given by

$$|\Psi(\theta)\rangle = \cos \theta |0^n\rangle + \sin \theta |1^n\rangle, \quad (1)$$

with  $0 \leq \theta \leq \pi/4$ . The GHZ state [18] is for  $\theta = \pi/4$ . However, K.Chen, S.Albeverio and S.-M.Fei [19] presented recently a family of Bell inequalities involving only two measurement settings per observer and show by an analytical proof that all the generalized GHZ entangled

states of  $n > 2$  qubits can violate the inequalities [20]. Such inequalities are highly desirable because they can lead to a much easier and more efficient way to test nonlocality, and contribute to the development of novel quantum protocols and cryptographic schemes by exploiting much less entangled resources and experimental efforts.

Here, we present a family of Bell-type inequalities, which are variants of the “standard” Bell inequalities. It is proved that all the generalized GHZ entangled states given by Eq.(1) can violate the inequalities. For  $n$  qubits, the quantum separability bound of our new inequalities is  $2^{n-1}$ , while the maximum possible entanglement violation is  $2^{(n-1)/2} + 2^{n-1}$ , which can be archived only by the GHZ state. Remarkably, the bound of local realism of the inequalities is 2, independent of  $n$  qubits. This yields that the local reality inequalities are exponentially stronger than the corresponding multipartite separability inequalities. Indeed, for  $n > 2$  qubits the inequalities show that separable quantum states do not satisfy the local reality inequalities, in contrast to the widespread opinion that any separable quantum state satisfies every classical probabilistic constraints. This is so because the inequalities are based on both mean values and variances of observables and hence exhibit effects of quantum fluctuation of multipartite systems. Thus, our new inequalities demonstrate nonclassical properties of separable quantum states and reveal that, the exponential violation of local realism by separable quantum states is an interesting consequence of quantum fluctuation of multipartite systems.

Let us give a brief review of MK Bell operators and the associated MK inequalities. The MK Bell operators of  $n$  qubits are defined recursively ( $n \geq 2$ ). Let  $\vec{a}_j \vec{\sigma}_j, \vec{a}'_j \vec{\sigma}_j$  denote spin observables on the  $j$ -th qubit,  $j = 1, \dots, n$ , where all  $\vec{a}_j, \vec{a}'_j$  are unit vectors in  $\mathbb{R}^3$  and  $\vec{\sigma}_j = (\sigma_x^j, \sigma_y^j, \sigma_z^j)$  is the Pauli matrices on the  $j$ -th qubit. Denote by  $\mathcal{B}_1 = \vec{a}_1 \vec{\sigma}_1$  and  $\mathcal{B}'_1 = \vec{a}'_1 \vec{\sigma}_1$ . Define

$$\mathcal{B}_n = \mathcal{B}_{n-1} \otimes \frac{1}{2}(\vec{a}_n \vec{\sigma}_n + \vec{a}'_n \vec{\sigma}_n) + \mathcal{B}'_{n-1} \otimes \frac{1}{2}(\vec{a}_n \vec{\sigma}_n - \vec{a}'_n \vec{\sigma}_n), \quad (2)$$

where  $\mathcal{B}'_n$  denotes the same expression  $\mathcal{B}_n$  but with all the  $\vec{a}_j$  and  $\vec{a}'_j$  exchanged.  $\mathcal{B}_n$  is called the MK Bell operator of  $n$  qubits. Assuming “local realism” [1, 2], one concludes the MK inequality of  $n$  qubits as follows:

$$\langle \mathcal{B}_n \rangle \leq 1, \quad (3)$$

which can be violated by quantum mechanics [9].

By convention, we adopt the notation  $|0^n\rangle = |0 \cdots 0\rangle$  and  $|1^n\rangle = |1 \cdots 1\rangle$ . Recall that

when all the  $\vec{a}_j$  and  $\vec{a}'_j$  are in the  $x - y$  plane and  $\vec{a}_j$ 's are distributed with angles  $(j - 1)(-1)^{n+1}2\pi/(2n)$  with respect to the  $x$ -axis and  $\vec{a}_j \perp \vec{a}'_j$ , the associated MK Bell operator has the following spectral decomposition [16]:

$$\mathcal{M}_n = 2^{(n-1)/2} (|\text{GHZ}_+\rangle\langle\text{GHZ}_+| - |\text{GHZ}_-\rangle\langle\text{GHZ}_-|), \quad (4)$$

where  $|\text{GHZ}_\pm\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle \pm |1^n\rangle)$  are GHZ's states [18]. Set

$$\mathcal{V}_n = \mathcal{M}_n + \mathcal{M}_n^2. \quad (5)$$

Then,

$$\langle\mathcal{V}_n\rangle = \langle\mathcal{M}_n\rangle + \langle\mathcal{M}_n\rangle^2 + \Delta(\mathcal{M}_n), \quad (6)$$

where  $\Delta(\mathcal{M}_n) = \langle(\mathcal{M}_n - \langle\mathcal{M}_n\rangle)^2\rangle$  is the variance of  $\mathcal{M}_n$  in a state.

*The separability inequalities for  $n$  qubits.*—For any separable quantum state of  $n$  qubits, one has

$$\langle\mathcal{V}_n\rangle \leq 2^{n-1}. \quad (7)$$

Indeed, note that  $\mathcal{V}_n = 2^{(n-1)/2}(|0^n\rangle\langle 1^n| + |1^n\rangle\langle 0^n|) + 2^{n-1}(|0^n\rangle\langle 0^n| + |1^n\rangle\langle 1^n|)$ . Then, for every product state  $|\psi\rangle = |\psi_1\rangle \cdots |\psi_n\rangle$  of  $n$  qubits, one has

$$\begin{aligned} \langle\mathcal{V}_n\rangle &= 2^{(n-1)/2}(\prod_{j=1}^n \alpha_j \beta_j^* + \prod_{j=1}^n \alpha_j^* \beta_j) \\ &\quad + 2^{n-1}(\prod_{j=1}^n |\alpha_j|^2 + \prod_{j=1}^n |\beta_j|^2) \\ &\leq 2^{(n-1)/2}(\prod_{j=1}^n |\alpha_j| + \prod_{j=1}^n |\beta_j|)^2 \\ &\quad + (2^{n-1} - 2^{(n-1)/2})(\prod_{j=1}^n |\alpha_j|^2 + \prod_{j=1}^n |\beta_j|^2), \end{aligned}$$

where  $\alpha_j = \langle\psi_j|0\rangle$  and  $\beta_j = \langle\psi_j|1\rangle$ ,  $j = 1, \dots, n$ . Using  $\max(x \sin \phi + y \cos \phi) = \sqrt{x^2 + y^2}$ , we get  $(\prod_{j=1}^n |\alpha_j| + \prod_{j=1}^n |\beta_j|)^2 \leq 1$  for  $n \geq 2$ . This concludes that Eq.(7) holds for all product states. Since a separable state is a convex combination of product states, it is concluded that Eq.(7) holds for all separable quantum states of  $n \geq 2$  qubits.

On the other hand, for every product state  $|\psi\rangle = |\psi_1\rangle \cdots |\psi_n\rangle$ , there is a local unitary transformation  $U = U_1 \otimes \cdots \otimes U_n$  such that  $U|\psi\rangle = |0^n\rangle$  and so,  $\langle\psi|U^\dagger \mathcal{V}_n U|\psi\rangle = 2^{n-1}$ , where  $U^\dagger$  denotes the adjoint operator of  $U$ . This yields that the equality of Eq.(7) can be archived by product states.

*The maximum possible entanglement violation.*—By Eq.(4), for any entangled state we have

$$\langle\mathcal{V}_n\rangle \leq 2^{(n-1)/2} + 2^{n-1}. \quad (8)$$

It is easy to check that for the generalized GHZ states  $|\Psi(\theta)\rangle$  of Eq.(1),

$$\langle\Psi(\theta)|\mathcal{V}_n|\Psi(\theta)\rangle = 2^{(n-1)/2} \sin 2\theta + 2^{n-1}. \quad (9)$$

Thus, all generalized GHZ entangled states violate Eq.(7) and, the GHZ state is the only state that violates Eq.(7) maximally because the GHZ state has been shown to be the only state that violates the MK inequality maximally [16, 21].

*The local reality inequalities.*—From the classical view of local realism, the values of  $\vec{a}_j\vec{\sigma}_j, \vec{a}'_j\vec{\sigma}_j$  are predetermined by a local hidden variable (LHV)  $\lambda$  before measurement, and independent of measurements, orientations or actions performed on other parties at spacelike separation. We denote by  $\varrho(\lambda)$  the statistical distribution of  $\lambda$  satisfying  $\varrho(\lambda) \geq 0$  and  $\int d\lambda \varrho(\lambda) = 1$ . Since  $-1 \leq \mathcal{M}_n(\lambda) \leq 1$  for the local hidden variable  $\lambda$ , one has

$$\langle\mathcal{M}_n\rangle_{\text{LHV}} = \int d\lambda \varrho(\lambda) \mathcal{M}_n(\lambda) \leq 1,$$

and

$$\begin{aligned} \Delta(\mathcal{M}_n)_{\text{LHV}} &= \int d\lambda \varrho(\lambda) [\mathcal{M}_n(\lambda) - \langle\mathcal{M}_n\rangle_{\text{LHV}}]^2 \\ &\leq 1 - \langle\mathcal{M}_n\rangle_{\text{LHV}}^2. \end{aligned}$$

Therefore, by Eq.(6) we have

$$\langle\mathcal{V}_n\rangle_{\text{LHV}} = \langle\mathcal{M}_n\rangle_{\text{LHV}} + \langle\mathcal{M}_n\rangle_{\text{LHV}}^2 + \Delta(\mathcal{M}_n)_{\text{LHV}} \leq 2. \quad (10)$$

Surprisingly, combining the separability inequality Eq.(7) and the local reality inequality Eq.(10) we conclude that separable quantum states of  $n > 2$  qubits do not satisfy the local reality inequalities, in contrast to the widespread opinion that any separable quantum state satisfies every classical probabilistic constraints. That separable quantum states do not satisfy the local reality inequalities has been pointed out by Loubenets [22], but her definition of classicality involved is narrower than the usual concept of LHV and consequently, Ref.[22] does not demonstrate nonclassical properties of separable states [23]. However, our analysis based on both the mean values and variances of observables does involve the usual sense of LHV and indicates that the violation of local realism by separable states is an interesting consequence of quantum fluctuation of multipartite systems.

As already demonstrated by Werner [24], testing for entanglement within quantum theory, and testing quantum mechanics against LHV theories are not equivalent. Indeed, as shown in Ref.[25], the physical origins of EPR's local realism and quantum entanglement

are different. For a multipartite system which, having interacted in past, are now spatially separated, EPR's local realism means that elements of physical reality for one subsystem should be independent of what is done with the others. In contrast, quantum entanglement refers only to quantum multipartite states, whether or not the individual subsystems are spatially separated. The violation of Bell's inequalities assuming EPR's local realism by suitable entangled states is therefore an interesting but indirect consequence of quantum entanglement. Further, our Bell-type inequalities show that quantum fluctuation of multipartite systems, even in product states, can exhibit quantum nonlocality against LHV theories based on EPR's local realism.

In summary, we have presented a family of Bell-type inequalities, which are constructed directly from the "standard" Bell inequalities involving two dichotomic observables per site. It is shown that the inequalities are violated by all the generalized GHZ entangled states of multiqubits. Remarkably, our new inequalities, based on both mean values and variances of observables, can provide stronger non-locality tests in a sense that the local reality inequalities are exponentially stronger than the corresponding multipartite separability inequalities. This reveals that the exponential violation of local realism by separable states is an interesting consequence of quantum fluctuation of multipartite systems. Complementary to the standard inequalities and a number of existing results, our result furthermore shed considerable light on LHV theories based on EPR's local realism. Evidently, the argument involved here can be generalized to all standard Bell inequalities. We hope that those variants of the usual Bell inequalities will play an important role in quantum information.

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- [1] J.S.Bell, Physics (Long Island City, N.Y.)**1**, 195(1964).
  - [2] A.Einstein, B.Podolsky, and N.Rosen, Phys. Rev.**47**, 777(1935).
  - [3] N.D.Mermin, Rev.Mod.Phys.**65**, 803(1993).
  - [4] N.Gisin, G.Ribordy, W.Tittle, and H.Zbinden, Rev. Mod. Phys.**74**, 145(2002); M.Nielsen and I.Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press,

- Cambridge, England, 2000).
- [5] Č.Brukner, M.Żukowski, and A.Zeilinger, Phys.Rev.Lett.**89**, 197901(2002).
  - [6] V.Scarani and N.Gisin, Phys.Rev.Lett.**87**, 117901(2001); A.Acín, N.Gisin, and V.Scarani, Quant.Inf.Compt.**3**, 563(2003); A.Acín, N.Gisin, and L.Masanes, Phys.Rev.Lett.**97**, 120405(2006).
  - [7] A search for “Bell inequalities” in tittle on [www.arxiv.org/quant-ph](http://www.arxiv.org/quant-ph) yields 290 results.
  - [8] J.F.Clauser, M.A.Horne, A.Shimony, R.A.Holt, Phys.Rev.Lett. **23**, 880(1969).
  - [9] N.D.Mermin, Phys.Rev.Lett. **65**, 1838(1990); S.M.Roy and V.Singh, Phys.Rev.Lett.**67**, 2761(1991); M.Ardehali, Phys.Rev.A **46**, 5373(1992); A.V.Belinskii, D.N.Klyshko, Phys. Usp. **36**, 653(1993); N.Gisin and H.Bechmann-Pasquinucci, Phys. Lett. A **246**, 1(1998).
  - [10] R.F.Werner and M.M.Wolf, Phys.Rev.A **64**, 032112(2001); M.Żukowski and Č.Brukner, Phys.Rev.Lett. **88**, 210401(2002).
  - [11] D.Kaszlikowski, *et al.*, Phys.Rev.Lett. **85**, 4418(2000); D.Collins, *et al.*, Phys.Rev.Lett. **88**, 040404(2002); X.-H.Wu and H.-S.Zong, Phys.Rev.A **68**, 032102(2003); L.B.Fu, Phys.Rev.Lett. **92**, 130404(2004); A.Acín, *et al.*, Phys.Rev.Lett. **92**, 250404(2004); W.Laskowski, T.Paterek, M.Żukowski, and Č.Brukner, Phys.Rev.Lett. **93**, 200401(2004); W.Son, J.Lee, and M.S.Kim, Phys.Rev.Lett. **96**, 060406(2006).
  - [12] J.Uffink, Phys.Rev.Lett. **88**, 230406(2002); S.Yu, Z.-B.Chen, J.-W.Pan, and Y.-D.Zhang, Phys.Rev.Lett. **90**, 080401(2003); J.Uffink and M.Seevinck, [quant-ph/0604145](http://arxiv.org/abs/quant-ph/0604145).
  - [13] Z.Chen, Phys.Rev.A **71**, 052302(2005); Phys.Rev.A **73**, 034306(2006).
  - [14] R.F.Werner and M.M.Wolf, Quantum Inf. Comp.**1**, 1(2001); M.Genovese, Phys.Rep.**413**, 319(2005); T.Paterek, W.Laskowski, and M.Żukowski, Mod.Phys.Lett.A **21**, 111(2006).
  - [15] N.Gisin, Phys.Lett. A **154**, 201(1991).
  - [16] V.Scarani and N.Gisin, J.Phys.A: Math.Gen.**34**, 6043(2001).
  - [17] M.Żukowski, Č.Brukner, W.Laskowski, and M.Wieśniak, Phys.Rev.Lett.**88**, 210402(2002).
  - [18] D.M.Greenberger, M.A.Horne, and A.Zeilinger, in *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M.Kafatos (Kluwer, Dordrecht, 1989), p.69; N.D.Mermin, Am. J. Phys. **58**, 731(1990).
  - [19] K.Chen, S.Albeverio, and S.-M.Fei, Two-setting Bell inequalities for many qubits, preprint.
  - [20] A Bell-type inequality for probabilities is presented in [J.L.Chen, C.Wu, L.C.Kwek, and C.H.Oh, Phys.Rev.Lett. **93**, 140407(2004)], which can be violated by all pure entangled states

of three qubits as shown by numerical results but without a rigorous proof.

- [21] Z.Chen, Phys.Rev.Lett. **93**, 110403(2004).
- [22] E.R.Loubenets, Phys.Rev.A **69**, 042102(2004).
- [23] C.Simon, Phys.Rev.A **69**, 026102(2005).
- [24] R.F.Werner, Phys.Rev.A **40**, 4277(1989).
- [25] S.M.Roy, Phys.Rev.Lett.**94**, 010402(2005).